

Majorization-preserving quantum channels

Lin Zhang^{*}

Institute of Mathematics, Hangzhou Dianzi University, Hangzhou 310018, PR China

Abstract

In this report, we give a characterization to those quantum channels that preserve majorization relationship between quantum states. Some remarks are presented as well.

1 Introduction

The notion of majorization has been widely used in many fields of mathematics, such as matrix analysis, operator theory, frame theory, and inequalities involving convex functions. The readers can be referred to a standard literature by Marshall and Olkin [1].

This report discusses the notion of majorization and some of its connections to quantum information. As a tool used in quantum information theory, the main application of majorization is first given by M. Nielsen [2]. His result precisely characterizes when it is possible for two parties to transform one pure state into another by means of local operations and classical communication (LOCC). There are other interesting applications of the notion can be found in [3, 4].

^{*}E-mail: godyalin@163.com; linyz@hdu.edu.cn

2 Preliminaries

Let \mathcal{H} be a finite dimensional complex Hilbert space. A *quantum state* ρ on \mathcal{H} is a positive semi-definite operator of trace one, in particular, for each unit vector $|\psi\rangle \in \mathcal{H}$, the operator $\rho = |\psi\rangle\langle\psi|$ is said to be a *pure state*. Now the set of all linear operators acting on \mathcal{H} denoted by $L(\mathcal{H})$. We denote by $\text{Herm}(\mathcal{H})$ the set of all Hermitian operators in $L(\mathcal{H})$. The notation $U(\mathcal{H})$ stands for the set of all unitary operator in $L(\mathcal{H})$. The set of all quantum states on \mathcal{H} is denoted by $D(\mathcal{H})$. For each quantum state $\rho \in D(\mathcal{H})$, its von Neumann entropy is defined by $S(\rho) = -\text{Tr}(\rho \log_2 \rho)$. A *quantum channel* Φ on \mathcal{H} is a trace-preserving completely positive linear map defined over the set $D(\mathcal{H})$. It follows from ([5, Prop. 5.2 and Cor. 5.5]) that there exists linear operators $\{K_\mu\}_\mu$ on \mathcal{H} such that $\sum_\mu K_\mu^\dagger K_\mu = \mathbb{1}$ and $\Phi = \sum_\mu \text{Ad}_{K_\mu}$, that is, for each quantum state ρ , we have the Kraus representation

$$\Phi(\rho) = \sum_\mu K_\mu \rho K_\mu^\dagger.$$

The *Choi-Jamiołkowski isomorphism* for a quantum channel $\Phi \in T(\mathcal{H})$ is defined as

$$J(\Phi) \stackrel{\text{def}}{=} \Phi \otimes \mathbb{1}_{L(\mathcal{H})}(\text{vec}(\mathbb{1}) \text{vec}(\mathbb{1})^\dagger),$$

where $\text{vec}(\mathbb{1}) \stackrel{\text{def}}{=} \sum_j |jj\rangle$. In fact, $J(\Phi) = \sum_\mu \text{vec}(K_\mu) \text{vec}(K_\mu)^\dagger$.

2.1 Majorization for real vectors

For two vectors $u, v \in \mathbb{R}^n$, u is *majorized* by v , denoted by $u \prec v$, if

$$\sum_{i=1}^k u_i^\downarrow \leq \sum_{i=1}^k v_i^\downarrow; k = 1, \dots, n$$

and

$$\sum_{i=1}^n u_i^\downarrow = \sum_{i=1}^n v_i^\downarrow,$$

where $u_1^\downarrow \geq \dots \geq u_n^\downarrow$ is the decreasing rearrangement of all components of the vector u . In [6], Hardy, Littlewood and Polya obtained the following characterization:

Proposition 2.1 ([6]). *Let $u, v \in \mathbb{R}^n$. u is majorized by v , i.e. $u \prec v$, if and only if $u = Bv$ for some $n \times n$ bi-stochastic matrix B .*

Recall that a matrix $B = [b_{ij}]$ is called *stochastic matrix* if

$$b_{ij} \geq 0 \quad \text{and} \quad \sum_{i=1}^n b_{ij} = 1.$$

Furthermore, if the stochastic matrix B satisfy the identity $\sum_{j=1}^n b_{ij} = 1$ as well, then it is called *bi-stochastic matrix*. Let us write $\text{Sym}(\Sigma)$, for instance $\Sigma = \{1, \dots, n\}$, to denote the set of one-to-one and onto functions of the form $\pi : \Sigma \rightarrow \Sigma$ (or, in other words, the *permutation* of Σ). For each $\pi \in \text{Sym}(\Sigma)$, we define a matrix $P_\pi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ as

$$(P_\pi)_{ij} = \begin{cases} 1, & \text{if } i = \pi(j); \\ 0, & \text{otherwise.} \end{cases}$$

It is clear that every permutation matrix is bi-stochastic, and that the set of bi-stochastic matrices is a convex set. The following famous theorem establishes that the bi-stochastic matrices are, in fact, given by the convex hull of the permutation matrices [5].

Proposition 2.2 (The Birkhoff-von Neumann theorem). *Let $B : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a linear map. Then B is a bi-stochastic matrix if and only if there exists a probability vector $p \in \mathbb{R}^{n!}$ such that*

$$B = \sum_{\pi \in \text{Sym}(\Sigma)} p(\pi) P_\pi,$$

where $\Sigma = \{1, \dots, n\}$.

We say that $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ preserves the majorization if $Tx \prec Ty$ whenever $x \prec y$, i.e.,

$$x \prec y \implies Tx \prec Ty.$$

In [7], Ando characterized the structure of linear map over \mathbb{R}^n that preserves the majorization between two vectors.

Proposition 2.3 ([7]). *Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a linear map. Then $Tu \prec Tv$, whenever $u \prec v$, if and only if one of the following conditions hold.*

(i) $Tu = \text{Tr}(u) a$ for some $a \in \mathbb{R}^n$.

(ii) $Tu = \alpha Pu + \beta \text{Tr}(u) e$ for some $\alpha, \beta \in \mathbb{R}$ and permutation $P : \mathbb{R}^n \rightarrow \mathbb{R}^n$.

Note that throughout this paper the trace of a vector means: $\text{Tr}(u) = \sum_j u_j$.

2.2 Majorization for Hermitian operators

We will now define an analogous notion of majorization for Hermitian operators. For Hermitian operators $A, B \in \text{Herm}(\mathcal{H})$, we say that A is *majorized* by B , denoted by $A \prec B$, if there exists a *mixed unitary channel* $\Phi \in \mathcal{T}(\mathcal{H})$ such that $A = \Phi(B)$, i.e.

$$A = \Phi(B) = \sum_j p_j U_j B U_j^\dagger, \quad (\forall j : p_j \geq 0; \sum_j p_j = 1).$$

Inspiration for this definition partly comes from the Birkhoff-von Neumann theorem, the following theorem gives an alternate characterization of this relationship that also connects it with majorization for real vectors.

Denote by $\lambda(X)$ the all eigenvalues of $X \in \text{Herm}(\mathcal{H})$ arranged as a vector. In particular, $\lambda^\downarrow(X)$ stands for the modified vector $\lambda(X)$ whose components are arranged in decreasing order.

Proposition 2.4. *Let $A, B \in \text{Herm}(\mathcal{H})$. Then $A \prec B$ if and only if $\lambda(A) \prec \lambda(B)$.*

Concerning linear maps on matrices, the structure of linear maps that preserve a quantity or property is a hot topic. In particular, for a quantum channel preserving majorization, it is essential for discussing the von Neumann entropy preservation of quantum channels. Denote by $\mathcal{T}(\mathcal{H})$ the set of all linear super-operators

$$\Phi : \mathcal{L}(\mathcal{H}) \rightarrow \mathcal{L}(\mathcal{H}).$$

Specifically, we say that $\Phi \in \mathcal{T}(\mathcal{H})$ preserves the majorization if $\Phi(A) \prec \Phi(B)$ whenever $A \prec B$ over $\text{Herm}(\mathcal{H})$, i.e.,

$$A \prec B \implies \Phi(A) \prec \Phi(B).$$

A similar result to Proposition 2.3 was obtained by Hiai in [8]. This result can be described as follows:

Proposition 2.5 ([8]). *Let $\Phi \in \mathcal{T}(\mathcal{H})$ be a linear super-operator. Then the following statements are equivalent:*

(i) Φ preserves majorization, that is, $\Phi(A) \prec \Phi(B)$ whenever $A \prec B$, where $A, B \in \text{Herm}(\mathcal{H})$;

(ii) either there exists an $A_0 \in \text{Herm}(\mathcal{H})$ such that

(a) $\Phi(X) = \text{Tr}(X) A_0$, for all $X \in \text{Herm}(\mathcal{H})$,

or there exist a unitary $U \in \mathcal{U}(\mathcal{H})$ and $\alpha, \beta \in \mathbb{R}$ such that Φ has one of the following forms:

(b) $\Phi(X) = \alpha UXU^\dagger + \beta \text{Tr}(X) \mathbb{1}$, for all $X \in \text{Herm}(\mathcal{H})$;

(b) $\Phi(X) = \alpha UX^\top U^\dagger + \beta \text{Tr}(X) \mathbb{1}$, for all $X \in \text{Herm}(\mathcal{H})$.

3 Main result

We are interested in the structure of completely positive linear maps (CP) on matrices [9] since the physical transformations are all represented by CP maps.

Consider the linear super-operator $\Lambda \in \mathcal{T}(\mathcal{H})$ defined as follows:

$$\Lambda(X) = \frac{1}{n} \text{Tr}(X) \mathbb{1}.$$

This is the *completely depolarizing channel*, which sends every density operator to the maximally mixed state $\frac{1}{n} \mathbb{1}$. The Choi-Jamiołkowski representation of Λ is

$$J(\Lambda) = \frac{1}{n} \mathbb{1} \otimes \mathbb{1}.$$

Now we can state our result:

Theorem 3.1. *Let $\Phi \in \mathcal{T}(\mathcal{H})$ be a quantum channel. Then $\Phi(\rho) \prec \Phi(\sigma)$, whenever $\rho \prec \sigma$ ($\rho, \sigma \in \mathcal{D}(\mathcal{H})$), if and only if one of the following conditions hold.*

(i) $\Phi(\rho) = \omega$ for some $\omega \in \mathcal{D}(\mathcal{H})$.

(ii) $\Phi(\rho) = \lambda U \rho U^\dagger + (1 - \lambda) \frac{1}{n} \mathbb{1}$ for some $\lambda \in \left[-\frac{1}{n^2-1}, 1\right]$ and unitary $U \in \mathcal{U}(\mathcal{H})$.

(iii) $\Phi(\rho) = \lambda U \rho^\top U^\dagger + (1 - \lambda) \frac{1}{n} \mathbb{1}$ for some $\lambda \in \left[-\frac{1}{n-1}, \frac{1}{n+1}\right]$ and unitary $U \in \mathcal{U}(\mathcal{H})$.

Proof. (\Leftarrow). This direction of the proof is trivially.

(\Rightarrow). From the Proposition 2.5, it follows that, either

$$\Phi(\rho) = \text{Tr}(\rho) A_0 = A_0.$$

Since Φ is a quantum channel, we have that $A_0 \geq 0$ and $\text{Tr}(A_0) = 1$. Setting $A_0 = \omega \in \mathcal{D}(\mathcal{H})$, we have $\Phi(\rho) = \omega$ for all $\rho \in \mathcal{D}(\mathcal{H})$. Or Φ must be of the following forms:

$$\Phi(\rho) = \alpha U \rho U^\dagger + \beta \mathbb{1} \quad \text{or} \quad \Phi(\rho) = \alpha U \rho^\top U^\dagger + \beta \mathbb{1}.$$

Let

$$\Psi(\rho) \equiv \alpha \rho + \beta \mathbb{1} \quad \text{or} \quad \Psi(\rho) \equiv \alpha \rho^\top + \beta \mathbb{1}.$$

Thus $\Phi(\rho) = U \Psi(\rho) U^\dagger$ and $\Psi = \alpha \mathbb{1}_{\mathcal{L}(\mathcal{H})} + \beta \Lambda$ or $\alpha T + \beta \Lambda$, where T is the transpose super-operator, defined by $T(\rho) = \rho^\top$. Therefore Φ is completely positive if and only if Ψ is completely positive. By the trace-preservation of Ψ , we have that $\alpha + n\beta = 1$. Moreover, we have the Choi-Jamiołkowski representation

$$J(\Psi) = \alpha \text{vec}(\mathbb{1}) \text{vec}(\mathbb{1})^\dagger + \beta \mathbb{1} \otimes \mathbb{1} \quad \text{or} \quad J(\Psi) = \alpha J(T) + \beta \mathbb{1} \otimes \mathbb{1}.$$

By the semi-definite positivity of $J(\Psi)$, we have

$$\begin{cases} n\alpha + \beta & \geq 0 \\ \beta & \geq 0 \end{cases} \quad \text{or} \quad \begin{cases} \alpha + \beta & \geq 0 \\ -\alpha + \beta & \geq 0 \end{cases}$$

The above inequalities can be solved via the following identification: $\alpha = \lambda$ and $\beta = \frac{1-\lambda}{n}$. Since the $J(T)$ is a self-adjoint and unitary, thus all eigenvalues of $J(T)$ are $\{\pm 1\}$. The desired conclusions can be obtained via some simple calculations. \square

4 Discussion

Recently, in [10], Li obtained the following result:

Proposition 4.1. *Let $\rho, \sigma \in \mathcal{D}(\mathcal{H})$. If $\rho \prec \sigma$ and $S(\rho) = S(\sigma)$, then $\lambda^\downarrow(\rho) = \lambda^\downarrow(\sigma)$.*

Recall that the von Neumann entropy of a quantum state ρ is defined as $S(\rho) = -\text{Tr}(\rho \log_2 \rho)$. This result indicates that, if $\rho \prec \sigma$ and $S(\rho) = S(\sigma)$, then there exists some unitary $U \in \mathcal{U}(\mathcal{H})$ such that $\rho = U \sigma U^\dagger$. Furthermore, we can have the following conclusion: if $\rho \prec \sigma$ and $\sigma \prec \rho$, denoted by $\rho \sim \sigma$, then $\rho = U \sigma U^\dagger$ for some unitary $U \in \mathcal{U}(\mathcal{H})$. Thus if a quantum channel preserves majorization, i.e. $\Phi(\rho) \prec \Phi(\sigma)$ whenever $\rho \prec \sigma$, then we can infer that $\Phi(\rho) \sim \Phi(\sigma)$ whenever $\rho \sim \sigma$. In other words,

$$\rho = U \sigma U^\dagger \implies \Phi(\rho) = V \Phi(\sigma) V^\dagger.$$

Finally, The characterization of majorization-preserving quantum channels can be reduced to the characterization of unitary equivalence preserving quantum channels.

5 Concluding remarks

In this report, we characterized those quantum channels that preserves the majorization. We found that majorization-preserving quantum channels are equivalently described as unitary equivalence preserving quantum channels. Some open problems are still left for the future research. For example,

What can be derived for $S(\rho) = S(\sigma)$, where $\rho, \sigma \in \mathcal{D}(\mathcal{H})$?

If we can solve this problem, then we can use it to make an analysis of classification of quantum states according to the amount of the encoded information in quantum states. We can make further analysis of the von Neumann entropy preserving quantum channels over \mathcal{H} : $S(\Phi(\rho)) = S(\rho)$. In particular, a special case, i.e. a unit-preserving quantum channel preserves the von Neumann entropy of a fixed quantum state, is characterized in [11]. Or we can still consider the characterization of quantum channel Φ such that

$$S(\rho) = S(\sigma) \implies S(\Phi(\rho)) = S(\Phi(\sigma)).$$

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References

- [1] A.W. Marshall and I. Olkin. Inequalities: Theory of Majorization and Its Application. Academic Press, New York (1979).
- [2] M. Nielsen and Guifr  Vidal. Majorization and the interconversion of bipartite states. Quant. Inf. & Comput. **1**(1), 76-93(2001).
- [3] T. Hiroshima. Majorization criterion for distillability of a bipartite quantum state. Phys. Rev. Lett. **91**, 057902 (2003).

- [4] M.H. Partovi. Entanglement detection using majorization uncertainty bounds. *Phys. Rev. A* **86**, 022309 (2012).
- [5] J. Watrous, *Theory of Quantum Information*, University of Waterloo, Waterloo (2008). See <http://www.cs.uwaterloo.ca/~watrous/quant-info/>
- [6] G.H. Hardy, J.E. Littlewood and G. Polya. *Inequalities* (2ed) Cambridge University Press (1932).
- [7] T. Ando. Doubly stochastic matrices, and comparison of eigenvalues. *Linear Algebra Appl.* **118**, 163-248 (1989).
- [8] F. Hiai. Similarity preserving linear maps on matrices. *Linear Algebra Appl.* **97**, 127-139 (1987).
- [9] G. Lindblad. Completely positive maps and entropy inequalities. *Commun. Math. Phys.* **40**, 147-151 (1975).
- [10] Y. Li and Y. Wang. Further results on entropy and separability. *J. Phys. A: Math. Theor.* **45**, 385305 (2012).
- [11] L. Zhang and J. Wu. Von Neumann entropy-preserving quantum operations. *Phys. Lett.* **375**(47): 4163-4165 (2011).